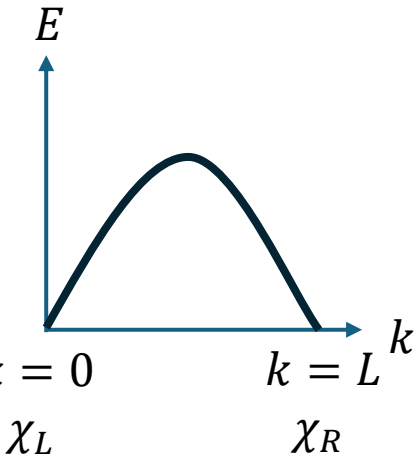
Non-invertible and chiral symmetries II

Shu-Heng Shao (MIT)

Croucher Summer Course, Emergent Symmetry

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UV and IR symmetries



- UV Kitaev chain

$$H = -i \sum_{\ell=1}^{2L} \chi_\ell \chi_{\ell+1}$$

- $G_{UV} = \mathbb{Z}_2 \times \mathbb{Z}_{2L}$

$$(-1)^F \chi_\ell (-1)^F = -\chi_\ell$$

$$T^{Maj} \chi_\ell \left(T^{Maj}\right)^{-1} = \chi_{\ell+1}$$

- IR Majorana fermion

$$\mathcal{L} = \frac{i}{2}\chi_L(\partial_t + \partial_x)\chi_L + \frac{i}{2}\chi_R(\partial_t - \partial_x)\chi_R$$

- $G_{IR} = \mathbb{Z}_2 \times \mathbb{Z}_2^R$

$$(-1)^F \chi_L (-1)^F = -\chi_L,$$
$$(-1)^F \chi_R (-1)^F = -\chi_R$$

$$(-1)^{F_R} \chi_L (-1)^{F_R} = \chi_L,$$
$$(-1)^{F_R} \chi_R (-1)^{F_R} = -\chi_R$$

$T^{Maj} = (-1)^{F_R} \exp(2\pi i \frac{P}{2L})$, P: continuum translation operator

Last time

Kitaev chain

$$H = -i\sum_{\ell=1}^{2L} \chi_\ell \chi_{\ell+1}$$

gauge $(-1)^F$ ⟷

Ising lattice model

$$H_{Ising} = -\sum_{j=1}^{L} X_j - \sum_{j=1}^{L} Z_j Z_{j+1}$$

symmetry $G_{UV} = \mathbb{Z}_2 \times \mathbb{Z}_{2L}$

$(-1)^F = i^N \chi_1 \chi_2 \cdots \chi_{2N}$

dual ⟷

symmetry

$\eta = X_1 X_2 \cdots X_L$

Maj lattice translation T^{Maj} ⟷

$$D = e^{2\pi i \frac{L}{8}} \left(\prod_{j=1}^{L-1} \frac{1 - i\,X_j}{\sqrt{2}} \frac{1 - i\,Z_j Z_{j+1}}{\sqrt{2}} \right) \frac{1 - i\,X_L}{\sqrt{2}} \frac{1+\eta}{\sqrt{2}}$$

$D^2 = (1+\eta)T$

Continuum Majorana CFT

$$\mathcal{L} = \frac{i}{2}\chi_L(\partial_t - \partial_x)\chi_L + \frac{i}{2}\chi_R(\partial_t + \partial_x)\chi_R$$

gauge $(-1)^F$ ⟷

Continuum Ising CFT

symmetry $G_{IR} = \mathbb{Z}_2 \times \mathbb{Z}_2^R$

$(-1)^F: \chi_L \to -\chi_L, \chi_R \to -\chi_R$

dual ⟷

symmetry

$\eta: \varepsilon \to \varepsilon, \sigma \to -\sigma$

$(-1)^{F_R}: \chi_L \to \chi_L, \chi_R \to -\chi_R$

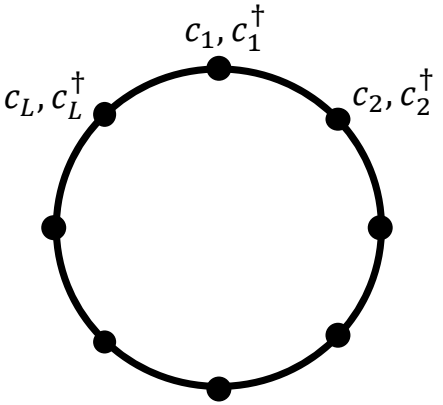
$i\chi_L\chi_R = \varepsilon$ ⟷

$\mathcal{D}\varepsilon = -\varepsilon\mathcal{D}$

$\mathcal{D}^2 = 1 + \eta$

Continuous Lattice Chiral Symmetry

Complex lattice fermion



- We place one complex fermion c_j on every lattice site j.

$$\{c_j, c_{j'}^\dagger\} = \delta_{jj'} \quad , \qquad \{c_j, c_{j'}\} = \{c_j^\dagger, c_{j'}^\dagger\} = 0$$

- Tight-binding Hamiltonian (known as the staggered fermion in high energy physics [Banks, Kogut, Susskind '75-'77]):

$$H = -i \sum_{j=1}^{L} (c_j^\dagger c_{j+1} + c_j c_{j+1}^\dagger)$$

- The continuum limit is <u>one</u> free, massless, non-chiral Dirac fermion.

Vector and axial symmetries

- A free, massless Dirac fermion $\Psi = \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix}$ in 1+1d:

$$\mathcal{L} = i\Psi_L^\dagger(\partial_t + \partial_x)\Psi_L + i\Psi_R^\dagger(\partial_t - \partial_x)\Psi_R$$

Ψ_L is a one-component, complex, left Weyl fermion, and similarly for Ψ_R.

$$\text{vector } U(1)^{\mathcal{V}}\text{: } \Psi_L \to e^{-i\theta}\Psi_L \quad , \quad \Psi_R \to e^{-i\theta}\Psi_R$$

$$\text{axial } U(1)^{\mathcal{A}}\text{: } \Psi_L \to e^{-i\theta}\Psi_L \quad , \quad \Psi_R \to e^{+i\theta}\Psi_R$$

- The global form of the group is $[U(1)^{\mathcal{V}} \times U(1)^{\mathcal{A}}]/\mathbb{Z}_2$.

U(1) symmetry on the lattice

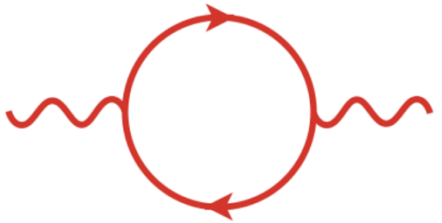
$$H = -i \sum_{j=1}^{L} (c_j^\dagger c_{j+1} + c_j c_{j+1}^\dagger)$$

- There is a manifest $U(1)^V$ symmetry, which flows to the vector symmetry of the continuum Dirac fermion theory.

$$U(1)^V: \; c_j \to e^{-i\theta} c_j$$

- Where is the axial $U(1)^A$ symmetry on the lattice?

Schwinger anomaly



- Common lore states that it's difficult to realize the axial U(1) symmetry on the lattice, because of an ('t Hooft) anomaly between the vector and axial global symmetries:

The axial symmetry is broken when the vector symmetry is gauged, and vice versa [Schwinger '59, Johnson '63].

$$\partial_\mu j^{A\mu} = \frac{1}{\pi} E$$

where $j^{A\mu}$ is the axial Noether current and E is the electric field.

- Perturbative anomalies are particularly challenging to realize on a lattice with finite-dim local Hilbert spaces (e.g., qubits). E.g., [Fidkowski-Xu '23].
- This anomaly can be realized on lattices with infinite-dim local Hilbert spaces [..., Sulejmanpasic-Gattringer '19, Gorantla-Lam-Seiberg-SHS '21, Cheng-Seiberg '22, Fazza-Sulejmanpasic '22,...]

Today [Chatterjee-Pace-SHS (PRL '25)]

$$H = -i \sum_{j=1}^{L} (c_j^\dagger c_{j+1} + c_j c_{j+1}^\dagger)$$

- We will realize both the vector and axial charges Q^V and Q^A on the lattice.
- $[Q^V, H] = [Q^A, H] = 0$ and $Q^V \in \mathbb{Z}$, $Q^A \in \mathbb{Z}$.
- BUT,

$$[Q^V, Q^A] \neq 0$$

Be real

- On the lattice, we decompose every complex fermion c_j into two real fermions a_j, b_j:

$$c_j = a_j + ib_j, \qquad a_j^\dagger = a_j \ , \qquad b_j^\dagger = b_j$$

$$\{a_j, a_{j'}\} = \{b_j, b_{j'}\} = 2\delta_{jj'}$$

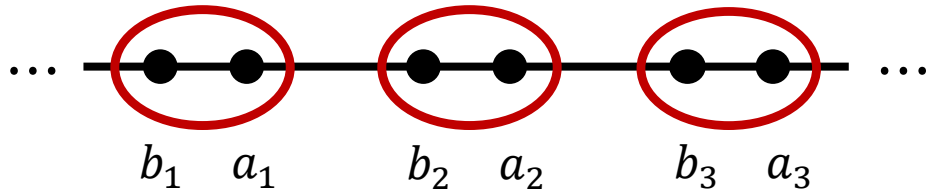
$$H = -2i \sum_{j=1}^{L} \left(a_j a_{j+1} + b_j b_{j+1}\right)$$

It has the Majorana translation symmetry

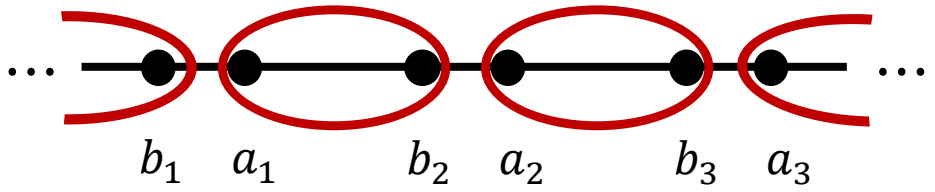
$$T^a : a_j \to a_{j+1}, \qquad b_j \to b_j$$

Symmetry = Topological Phase

$$Q^V = \frac{i}{2}\sum_j a_j b_j$$

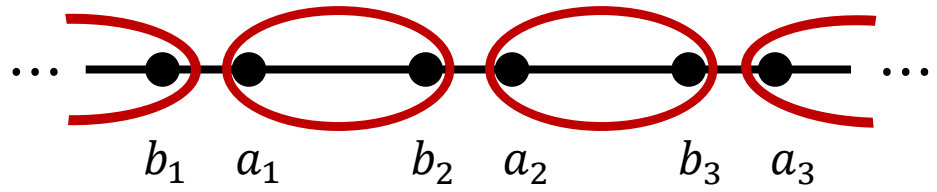


$$Q^A = (T^a)^{-1} Q^V T^a$$
$$= \frac{i}{2}\sum_j a_{j-1} b_j$$



Axial charge = Kitaev chain

Lattice axial charge



$$Q^A = \frac{i}{2}\sum_j a_j b_{j+1} = \frac{1}{2}\sum_j (c_j + c_j^\dagger)(c_{j+1} - c_{j+1}^\dagger)$$

1. It commutes with the tight-binding Hamiltonian, $[Q^A, H] = 0$.
2. It is quantized, i.e., it has integer eigenvalues, $Q^A \in \mathbb{Z}$. Hence, $e^{i\theta Q^A}$ generates a compact $U(1)$ global symmetry.
3. It is not on-site but locality-preserving:

$$e^{i\theta Q^A} c_j e^{-i\theta Q^A} = \cos\theta\, c_j - \frac{i}{2}\sin\theta\,(c_{j-1}^\dagger + c_{j-1} - c_{j+1}^\dagger + c_{j+1})$$

But...

$$[Q^V, Q^A] \neq 0 \text{ on the lattice}$$

$$[Q^V, Q^A] = 0 \text{ in the continuum}$$

Non-abelian algebra $[Q^V, Q^A] \neq 0$

- What algebra do these quantized charges Q^V, Q^A form?
- Define $(n = 1,2, \cdots, L)$

$$Q_n = \frac{i}{2}\sum_j a_j b_{j+n}\,, \qquad G_n = \frac{i}{2}\sum_j (a_j a_{j+n} - b_j b_{j+n})$$

where $Q_0 = Q^V, Q_1 = Q^A$.

- They generate a non-abelian algebra

$$[Q_n, Q_m] = iG_{m-n}\ , \qquad [G_n, G_m] = 0$$
$$[Q_n, G_m] = 2i(Q_{n-m} - Q_{n+m})$$

This algebra was first discovered by Onsager in 1944. See [Vernier-O'Brien-Fendley '18] for recent discussions.

Onsager algebra as lattice chiral symmetry

$$Q_n = \frac{i}{2}\sum_j a_j b_{j+n}\,, \qquad G_n = \frac{i}{2}\sum_j (a_j a_{j+n} - b_j b_{j+n})$$

- We start with $Q_0 = Q^V$ and $Q_1 = Q^A$ that are locality-preserving, but their nested commutators involve more and more non-local charges.
- For fixed n

$$\lim_{L\to\infty} Q_n = \begin{cases} Q^V\,, & n \in 2\mathbb{Z} \\ Q^A\,, & n \in 2\mathbb{Z}+1 \end{cases}$$

$$\lim_{L\to\infty} G_n = 0$$

- The continuum $[U(1)_V \times U(1)_A]/\mathbb{Z}_2$ abelian chiral symmetry arises from the lattice non-abelian Onsager algebra generated by Q^V, Q^A.

A tale of two charges

Compact $U(1)$

$$Q^A = \frac{i}{2}\sum_j a_j b_{j+1}$$

$$[Q^A, Q^V] \neq 0 \quad , \qquad Q^A \in \mathbb{Z}$$

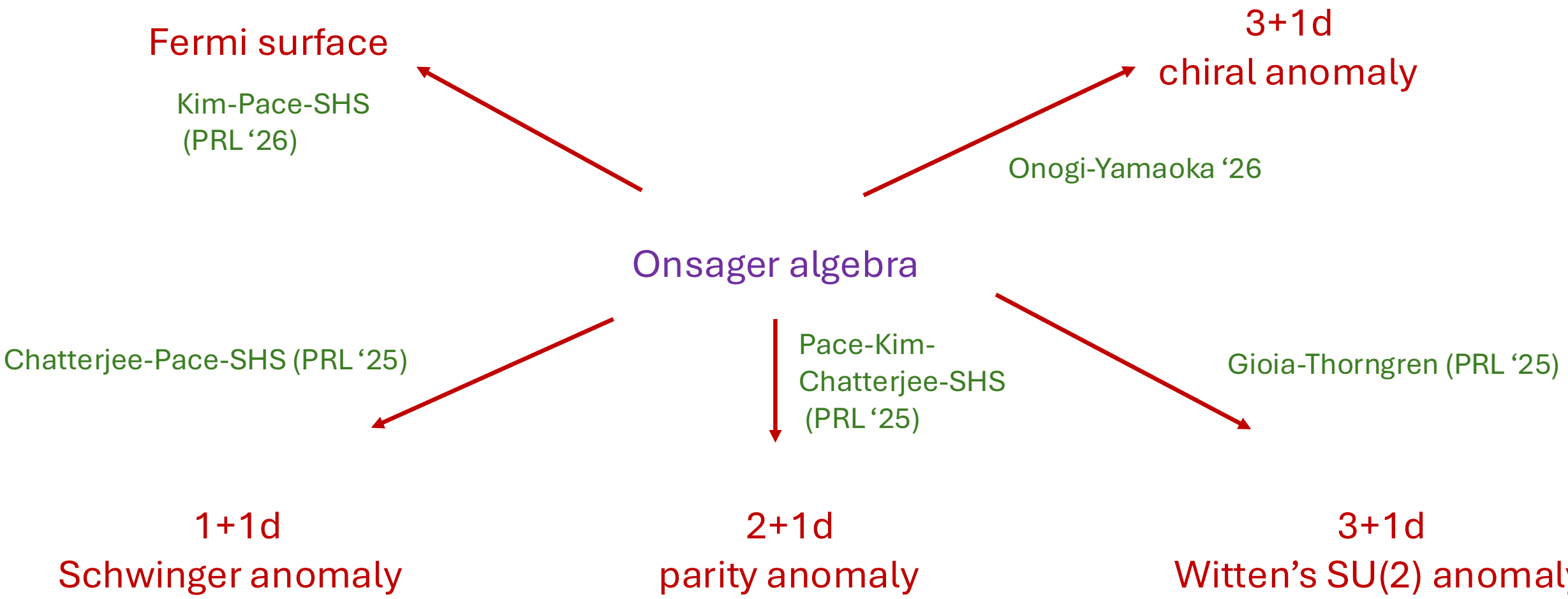
Non-compact $\mathbb{R}$

$$\tilde{Q}^A = \frac{i}{4}\sum_j (a_j b_{j+1} - b_j a_{j+1})$$

$$\left[\tilde{Q}^A, Q^V\right] = 0 \quad , \qquad \tilde{Q}^A \notin \mathbb{Z}$$

- Both Q^A and $\tilde{Q}^A$ flow to the same axial charge in the continuum limit.
- We do not have a lattice axial charge that is both (1) quantized and (2) commutes with the vector charge.
- (1)+(2) would have violated [Nielsen-Ninomiya '81][Sopenko-Kapustin '24].

Onsager rules them all



Non-invertible Symmetry in the XX model

XX = free fermion + $\mathbb{Z}_2$ gauge field

- UV free fermion

$$H = -i\sum_{j=1}^{L}(c_j^\dagger c_{j+1} + c_j c_{j+1}^\dagger)$$

G_{UV} = Onsager symmetry

- IR Dirac fermion

$$\mathcal{L} = i\Psi_L^\dagger(\partial_t + \partial_x)\Psi_L + i\Psi_R^\dagger(\partial_t - \partial_x)\Psi_R$$

$$G_{IR} = [U(1)_V \times U(1)_A]/\mathbb{Z}_2$$

- UV XX model [Lieb-Schultz-Mattis '61]

$$H_{XX} = \sum_{j=1}^{L} X_j X_{j+1} + Y_j Y_{j+1}$$

$$U(1)_M\colon\ Q_M = \frac{1}{2}\sum_j Z_j$$

- IR c=1 compact boson

$$\mathcal{L} = \frac{R^2}{4\pi}\left(\partial_\mu \phi\right)^2, R = \sqrt{2}$$

$$G_{IR} = U(1)_M \times U(1)_W$$

$$j_\mu^M = \frac{iR^2}{2\pi}\partial_\mu \phi, \qquad j_\mu^W = \frac{\epsilon_{\mu\nu}}{2\pi}\partial^\nu \phi$$

Non-invertible symmetry of the XX model

$$H_{XX} = \sum_{j=1}^{L} X_j X_{j+1} + Y_j Y_{j+1}$$

- It is related to the free lattice fermion model by gauging $(-1)^F$.
- The Majorana translation T^a of the fermion model becomes a non-invertible symmetry D^a of the XX model [Vernier-O'Brien-Fendley '18, Pace-Chatterjee-SHS '24].

$$D^a Z_j = \begin{cases} X_j Y_{j+1} D^a, & j: even \\ -Y_j X_{j+1} D^a, & j: odd \end{cases}$$

$$D^a X_j X_{j+1} = \begin{cases} X_j X_{j+1} D^a, & j: even \\ Y_{j+1} Y_{j+2} D^a, & j: odd \end{cases}, \quad D^a Y_j Y_{j+1} = \begin{cases} X_{j+1} X_{j+2} D^a, & j: even \\ Y_j Y_{j+1} D^a, & j: odd \end{cases}$$

Lattice winding symmetry

- The XX model has a manifest U(1) symmetry

$$Q_M = \frac{1}{2}\sum_j Z_j$$

- The non-invertible symmetry reveals a second lattice U(1) symmetry of the XX model:

$$D^a Q_M = 2Q_W D^a$$

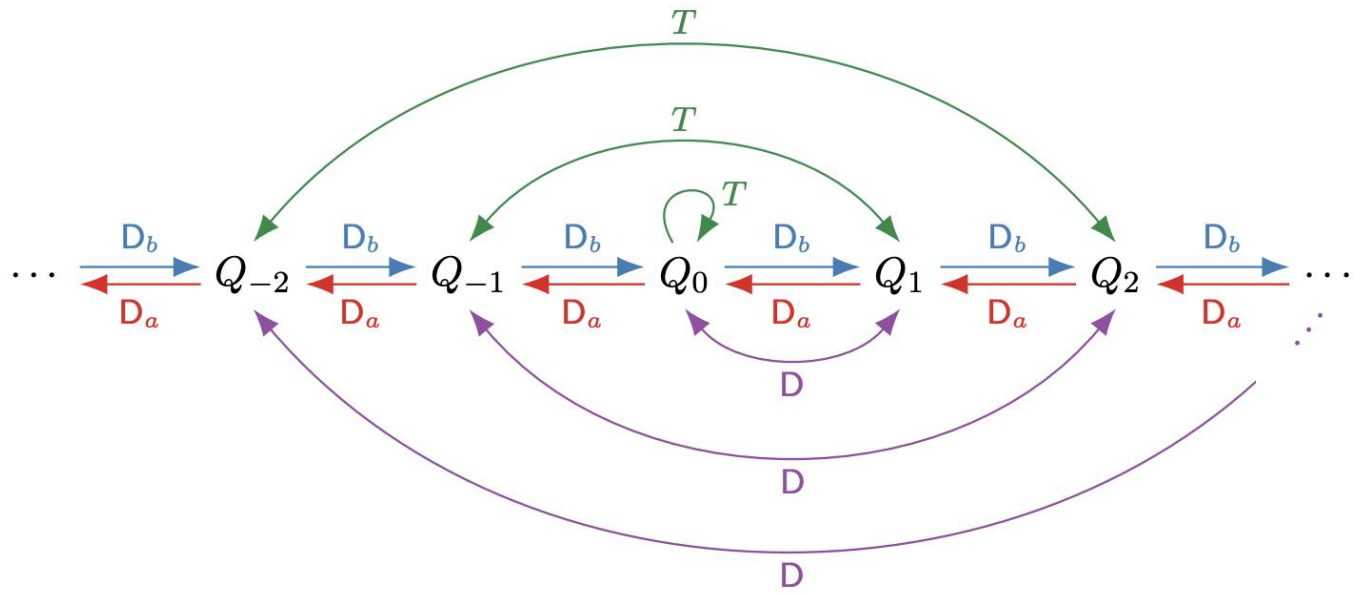
- Lattice winding charge

$$Q_W = \frac{1}{4}\sum_{n=1}^{L/2} X_{2n}Y_{2n+1} - Y_{2n-1}X_{2n}$$

- Again, Q_M and Q_W do not commute and form an Onsager algebra.

Onsager and non-invertible symmetry

[Jone-Prakash-Fendley '24, Pace-Chatterjee-SHS '25]



$$Q_n \to \begin{cases} 2Q_W, & n: odd \\ Q_M, & n: even \end{cases}$$

Figure 3: The symmetry operators T, D_a, D_b, and D of the XX model act nontrivially on the conserved Onsager charges Q_n. The color-coded arrows in this diagram describe the action on Q_n by the symmetry operators labeling them.

Gapless constraint from non-invertible symmetry

$$H = \sum_{j=1}^{L} X_j X_{j+1} + Y_j Y_{j+1} + \cdots$$

- There are infinitely many deformations to the XX Hamiltonian preserving

1. Non-invertible symmetry D^a, and
2. The onsite $U(1)_M$ symmetry $Q_M = \frac{1}{2}\sum_j Z_j$.

- These two symmetries imply that any symmetric Hamiltonian has to be gapless. [Ando-Ohmori '26]
- Similar gapless constraint from the Onsager algebra [Pace-Chatterjee-SHS '25].

Other topics

- LSM-type constraints: anomalies of non-invertible symmetries. E.g., constraints on RG flows, deconfinement in 1+1d adjoint QCD, …
- Non-invertible symmetries in higher dimensions: non-invertible symmetries in 3+1d QED, 3+1d Wegner duality symmetry, 2+1d examples, …
- Non-invertible SPT phases.
- Applications in particle physics: axions.
- Selection rules for scattering amplitudes.
- …